NEW INDEX TRANSFORMS WITH THE PRODUCT OF BESSEL FUNCTIONS

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ABSTRACT. New index transforms are investigated, which contain as the kernel products of the Bessel and modified Bessel functions. Mapping properties and invertibility in Lebesgue spaces are studied for these operators. Relationships with the Kontorovich-Lebedev and Fourier cosine transforms are established. Inversion theorems are proved. As an application, a solution of the initial value problem for the fourth order partial differential equation, involving the Laplacian is presented.

1. INTRODUCTION AND PRELIMINARY RESULTS

The main objects of the present paper are the following operators of the index transforms [1]

$$(Ff)(\tau) = \frac{1}{\sinh(\pi\tau/2)} \int_0^\infty K_{i\tau}(2\sqrt{2x})\ \mathrm{Im}\left[J_{i\tau}(2\sqrt{2x})\right] f(x)dx, \quad \tau \in \mathbb{R}\backslash\{0\}, \tag{1.1}$$

$$(Gg)(x) = \int_{-\infty}^\infty K_{i\tau}(2\sqrt{2x})\ \mathrm{Im}\left[J_{i\tau}(2\sqrt{2x})\right] \frac{g(\tau)}{\sinh(\pi\tau/2)}\, d\tau, \quad x \in \mathbb{R}_+. \tag{1.2}$$

Here i is the imaginary unit and Im denotes the imaginary part of a complex-valued function. The convergence of the integrals (1.1), (1.2) will be clarified below. Functions $J_\mu(z), K_\mu(z),\ \mu \in \mathbb{C}$ [2], Vol. II are, correspondingly, the Bessel and modified Bessel or Macdonald functions, which satisfy the differential equations

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} + (z^2-\mu^2)u = 0, \tag{1.3}$$

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} - (z^2+\mu^2)u = 0, \tag{1.4}$$

respectively. It has the asymptotic behaviour

$$J_\mu(z) = \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi}{4}(2\mu+1)\right)[1+O(1/z)], \quad K_\mu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}[1+O(1/z)], \qquad z\to\infty, \tag{1.5}$$

and near the origin

$$J_\mu(z) = O(z^\mu), \quad z^\mu K_\mu(z) = 2^{\mu-1}\Gamma(\mu) + o(1),\ z\to 0, \tag{1.6}$$

$$K_0(z) = -\log z + O(1),\ z\to 0. \tag{1.7}$$

The Macdonald function can be represented by the integral

$$K_\mu(z) = \int_0^\infty e^{-z\cosh u}\cosh(\mu u)du,\ \mathrm{Re}\, z>0,\ \mu\in\mathbb{C}. \tag{1.8}$$

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Concerning the kernel of the transform (1.1) our key representation will be in terms of the Mellin-Barnes integral, giving by relation (8.4.23.11) in [3], Vol. III, namely,

$$\frac{K_{i\tau}(2\sqrt{2x})}{\sinh(\pi\tau/2)} \operatorname{Im}\left[J_{i\tau}(2\sqrt{2x})\right] = -\frac{1}{16\pi i\sqrt{\pi}} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s/2)\Gamma((s+i\tau)/2)\Gamma((s-i\tau)/2)}{\Gamma((1-s)/2)} x^{-s} ds, \tag{1.9}$$

where $x > 0,\ \tau \in \mathbb{R}\backslash\{0\},\ \gamma > 0,\ \Gamma(z)$ is Euler's gamma -function [2], Vol. I. The theory of the index transforms can be found in the author monograph [1] (see also [4]). The familiar example is the Kontorovich-Lebedev transform

$$(KLf)(\tau) = \int_0^\infty K_{i\tau}(x) f(x) dx. \tag{1.10}$$

Our method of investigation is based on the theory of the Mellin transform in the Lebesgue spaces [5]. In fact, we define the Mellin transform as

$$f^*(s) = \int_0^\infty f(x) x^{s-1} dx \tag{1.11}$$

and its inverse, accordingly,

$$f(x) = \frac{1}{2\pi i} \int_{\nu-i\infty}^{\nu+i\infty} f^*(s) x^{-s} ds. \tag{1.12}$$

Integrals (1.11), (1.12) are convergent, for instance, in mean in the weighted L_p-spaces, $1 < p \le 2$ and the Parseval equality takes place [5]

$$\int_0^\infty f(x) g(x) dx = \frac{1}{2\pi i} \int_{\nu-i\infty}^{\nu+i\infty} f^*(s) g^*(1-s) ds. \tag{1.13}$$

It is important for us for further investigation to obtain an integral representation of the kernel in (1.1) in terms of the Fourier cosine transform [5]

$$(F_c f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos(xt) dt. \tag{1.14}$$

Precisely, we prove the following

Lemma 1. *Let* $x > 0, \tau \in \mathbb{R}\backslash\{0\}$. *Then*

$$\frac{K_{i\tau}(2\sqrt{2x})}{\sinh(\pi\tau/2)} \operatorname{Im}\left[J_{i\tau}(2\sqrt{2x})\right] = -\frac{2}{\pi} \int_0^\infty \cos(\tau u) \operatorname{Re}\left[K_0\left(4e^{\pi i/4}\sqrt{x\cosh u}\right)\right] du, \tag{1.15}$$

where Re *denotes the real part of a complex-valued function.*

Proof. Indeed, taking the Fourier cosine transform (1.14) from both sides of the equality (1.9), we change the order of integration owing to Fubini's theorem. In fact, employing the reciprocal equalities (see formula (1.104) in [1]) via the Fourier cosine transform (1.14)

$$\int_0^\infty \Gamma\left(\frac{s}{2}+\frac{i\tau}{2}\right)\Gamma\left(\frac{s}{2}-\frac{i\tau}{2}\right) \cos(u\tau) d\tau = \frac{\pi}{2^{s-1}} \frac{\Gamma(s)}{\cosh^s u}, \ \operatorname{Re} s > 0, \tag{1.16}$$

$$\Gamma\left(\frac{s}{2}+\frac{i\tau}{2}\right)\Gamma\left(\frac{s}{2}-\frac{i\tau}{2}\right) = \frac{\Gamma(s)}{2^{s-2}} \int_0^\infty \frac{\cos(\tau u)}{\cosh^s u} du, \tag{1.17}$$

one can integrate twice by parts in the integral (1.17), showing the uniform estimate

$$\left|\Gamma\left(\frac{s}{2}+\frac{i\tau}{2}\right)\Gamma\left(\frac{s}{2}-\frac{i\tau}{2}\right)\right| \le \frac{|\Gamma(s+1)|}{\tau^2}\left[c_1 + c_2|s|\,\right], \ \operatorname{Re} s > 0,\ \tau \in \mathbb{R}\backslash\{0\}, \tag{1.18}$$

where c_1, c_2 are absolute positive constants. Hence with the use of the Stirling asymptotic formula for the gamma-function [2], Vol. I and the elementary inequality

$$\left|\Gamma\left(\frac{s}{2}+\frac{i\tau}{2}\right)\Gamma\left(\frac{s}{2}-\frac{i\tau}{2}\right)\right| \le |\Gamma(s)|B(\gamma/2,\gamma/2) \tag{1.19}$$

where $B(a,b)$ is Euler's beta-function [2], Vol. I, it guarantees the absolute convergence of the iterated integral

$$\int_0^\infty \int_{\gamma-i\infty}^{\gamma+i\infty} \left|\frac{\Gamma(s/2)\Gamma((s+i\tau)/2)\Gamma((s-i\tau)/2)}{\Gamma((1-s)/2)} ds\right| d\tau < \infty.$$

Therefore, (1.9) and (1.16) yield

$$\int_0^\infty \frac{\cos(\tau u)}{\sinh(\pi\tau/2)} K_{i\tau}(2\sqrt{2x})\ \mathrm{Im}\left[J_{i\tau}(2\sqrt{2x})\right] d\tau = -\frac{1}{8i\sqrt{\pi}} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\Gamma(s/2)}{\Gamma((1-s)/2)} (2x\cosh u)^{-s} ds. \tag{1.20}$$

Moreover, appealing to the duplication formula for the gamma-function [2], Vol. I and making a simple substitution, the latter Mellin-Barnes integral can be written as

$$-\frac{1}{8i\sqrt{\pi}} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\Gamma(s/2)}{\Gamma((1-s)/2)} (2x\cosh u)^{-s} ds = -\frac{1}{8\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma^2(s)\Gamma(s+1/2)}{\Gamma(1/2-s)} (x^2\cosh^2 u)^{-s} ds. \tag{1.21}$$

Meanwhile, considering $0<\gamma<1/2$, we have the value of the relatively convergent integral (cf. [5], Section 7.3)

$$\frac{\Gamma(s)}{\Gamma(1/2-s)} = \frac{2^{1-2s}}{\sqrt{\pi}} \int_0^\infty \cos t\ t^{2s-1} dt = -\frac{2^{1-2s}(2s-1)}{\sqrt{\pi}} \int_0^\infty \sin t\ t^{2(s-1)} dt,$$

where the latter integral is obtained after the integration by parts and, evidently, converges absolutely for $0<\gamma<1/2$. Hence, substituting it in the right-hand side of (1.21) and changing the order of integration, we find with the aid of relation (8.4.23.1) in [3], Vol. III

$$-\frac{1}{8\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma^2(s)\Gamma(s+1/2)}{\Gamma(1/2-s)} (x^2\cosh^2 u)^{-s} ds$$

$$= \frac{1}{4\pi i\sqrt{\pi}} \int_0^\infty \frac{\sin t}{t^2} \int_{\gamma-i\infty}^{\gamma+i\infty} [2\Gamma(1+s)\Gamma(s+1/2) - \Gamma(s)\Gamma(s+1/2)] \left(\frac{2x\cosh u}{t}\right)^{-2s} ds dt$$

$$= \frac{1\sqrt{2x\cosh u}}{\sqrt{\pi}} \int_0^\infty K_{1/2}\left(\frac{4x\cosh u}{t}\right) \left[\frac{4x\cosh u}{t} - 1\right] \frac{\sin t}{t^{5/2}} dt.$$

Since

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z},$$

we get with the simple substitution

$$-\frac{1}{8\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma^2(s)\Gamma(s+1/2)}{\Gamma(1/2-s)} (x^2\cosh^2 u)^{-s} ds = \frac{1}{2} \int_0^\infty e^{-4x\cosh(u)/t} \left[\frac{4x\cosh u}{t} - 1\right] \frac{\sin t}{t^2} dt.$$

We calculate the latter integral, appealing to relation (2.5.37.1) in [3], Vol. I. In fact, taking in mind the identity for Macdonald's functions [2], Vol. II

$$K_{\mu+1}(z) - K_{\mu-1}(z) = \frac{2\mu}{z} K_\mu(z),$$

we deduce

$$\frac{1}{2} \int_0^\infty e^{-4x\cosh(u)/t} \left[\frac{4x\cosh u}{t} - 1\right] \frac{\sin t}{t^2} dt = -\frac{1}{2}\left[K_2\left(4e^{\pi i/4}\sqrt{x\cosh u}\right) + K_2\left(4e^{-\pi i/4}\sqrt{x\cosh u}\right)\right]$$

$$+\frac{1}{4i\sqrt{x\cosh u}}\left[e^{\pi i/4}K_1\left(4e^{\pi i/4}\sqrt{x\cosh u}\right)-e^{-\pi i/4}K_1\left(4e^{-\pi i/4}\sqrt{x\cosh u}\right)\right]$$
$$=-\mathrm{Re}\left[K_0\left(4e^{\pi i/4}\sqrt{x\cosh u}\right)\right].$$

Thus combining with (1.20), (1.21), we find the value of the index integral

$$\int_0^\infty \frac{\cos(\tau u)}{\sinh(\pi\tau/2)}K_{i\tau}(2\sqrt{2x})\ \mathrm{Im}\left[J_{i\tau}(2\sqrt{2x})\right]d\tau=-\mathrm{Re}\left[K_0\left(4e^{\pi i/4}\sqrt{x\cosh u}\right)\right],\ x>0,\ u\in\mathbb{R}. \tag{1.22}$$

In the meantime, using (1.9), (1.18), one can verify that for each $x>0$ the kernel of (1.1) belongs to $L_1(\mathbb{R}_+)\cap L_2(\mathbb{R}_+)$. Since (see (1.5), (1.7), (1.8))

$$\left|K_0\left(4e^{\pm\pi i/4}\sqrt{x\cosh u}\right)\right|=\left|\int_0^\infty \exp\left(-4e^{\pm\pi i/4}\sqrt{x\cosh u}\ \cosh t\right)dt\right|$$
$$\le\int_0^\infty \exp\left(-2\sqrt{2x\cosh u}\ \cosh t\right)dt=K_0(2\sqrt{2x\cosh u}),$$

the same is true for the right-hand side of (1.22). Hence taking the inverse Fourier cosine transform [5], we come up with equality (1.15), completing the proof of Lemma 1. □

Corollary 1. *It has*

$$\lim_{\tau\to 0}\ \frac{K_{i\tau}(2\sqrt{2x})}{\sinh(\pi\tau/2)}\mathrm{Im}\left[J_{i\tau}(2\sqrt{2x})\right]=-\frac{2}{\pi}\int_0^\infty \mathrm{Re}\left[K_0\left(4e^{\pi i/4}\sqrt{x\cosh u}\right)\right]du,\ x>0.$$

Corollary 2. *Let* $x>0,\tau\in\mathbb{R}\backslash\{0\}$. *Then the kernel in* (1.1) *satisfies the following inequality*

$$\left|\frac{K_{i\tau}(2\sqrt{2x})}{\sinh(\pi\tau/2)}\ \mathrm{Im}\left[J_{i\tau}(2\sqrt{2x})\right]\right|\le\frac{4}{\pi}\ e^{-\delta|\tau|}\ K_0^2\left(\cos\left(\frac{\delta}{2}\right)\sqrt{2x\cos(\delta)}\right), \tag{1.23}$$

where $\delta\in[0,\pi/2)$.

Proof. Indeed, the integral in the right-hand side of (1.15) can be written as

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\tau u}\left[K_0\left(4e^{\pi i/4}\sqrt{x\cosh u}\right)+K_0\left(4e^{-\pi i/4}\sqrt{x\cosh u}\right)\right]du$$

and by the analytic property of the integrand and the absolute convergence of the integral one can move the contour along the open infinite horizontal strip $(i\delta-\infty,\ i\delta+\infty)$ with $\delta\in[0,\pi/2)$, i.e.

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\tau u}\ \left[K_0\left(4e^{\pi i/4}\sqrt{x\cosh u}\right)+K_0\left(4e^{-\pi i/4}\sqrt{x\cosh u}\right)\right]du$$
$$=-\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\tau(i\delta+u)}\ \left[K_0\left(4e^{\pi i/4}\sqrt{x\cosh(i\delta+u)}\right)+K_0\left(4e^{-\pi i/4}\sqrt{x\cosh(i\delta+u)}\right)\right]du, \tag{1.24}$$

where the main branch of the square root is chosen. Hence denoting by

$$z=\cosh(i\delta+u)=\cos(\delta)\cosh u+i\sin(\delta)\sinh u=|z|e^{i\arg z},$$

where $\arg z=\arctan(\tan(\delta)\tanh u)\in(-\delta,\ \delta)$. Then

$$\mathrm{Re}\left[4\ e^{\pm\pi i/4}\sqrt{x\cosh(i\delta+u)}\right]=4\sqrt{x|z|}\cos\left(\frac{\arg z}{2}\pm\frac{\pi}{4}\right)\ge 4\sqrt{x\cos(\delta)\cosh u}\ \cos\left(\frac{\arg z}{2}\pm\frac{\pi}{4}\right)$$

when $\cos\left(\frac{\arg z}{2}\pm\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}\left[\cos(\arg z/2)\mp\sin(\arg z/2)\right]>0$, i.e $|\tan(\arg z/2)|<1$, which is true. Moreover, we have

$$\cos\left(\frac{\arg z}{2}\pm\frac{\pi}{4}\right)>\cos\left(\frac{\delta}{2}+\frac{\pi}{4}\right).$$

Therefore, returning to (1.15), (1.24), we obtain

$$\left|\frac{K_{i\tau}(2\sqrt{2x})}{\sinh(\pi\tau/2)}\ \mathrm{Im}\left[J_{i\tau}(2\sqrt{2x})\right]\right| \le \frac{2e^{-\delta|\tau|}}{\pi}\int_0^\infty K_0\left(4\cos\left(\frac{\delta}{2}+\frac{\pi}{4}\right)\sqrt{x\cos(\delta)\cosh u}\right)du$$

$$\le \frac{2e^{-\delta|\tau|}}{\pi}\int_0^\infty K_0\left(4\cos\left(\frac{\delta}{2}+\frac{\pi}{4}\right)\sqrt{x\cos(\delta)}\cosh(u/2)\right)du$$

$$= \frac{4e^{-\delta|\tau|}}{\pi}\int_0^\infty\int_0^\infty \exp\left(-4\cos\left(\frac{\delta}{2}+\frac{\pi}{4}\right)\sqrt{x\cos(\delta)}\cosh u\ \cosh t\right)du\,dt$$

$$\le \frac{4e^{-\delta|\tau|}}{\pi}K_0^2\left(\cos\left(\frac{\delta}{2}\right)\sqrt{2x\cos(\delta)}\right),$$

completing the proof of Corollary 2.

□

An immediate consequence of Lemma 1 is also

Corollary 3. *Let $x > 0$. Then for all $\tau \in \mathbb{R}$ the inequality*

$$\left|\frac{K_{i\tau}(2\sqrt{2x})}{\sinh(\pi\tau/2)}\ \mathrm{Im}\left[J_{i\tau}(2\sqrt{2x})\right]\right| \le \frac{4}{\pi}\,K_0^2\left(\sqrt{2x}\right) \tag{1.25}$$

is fulfilled.

Proof. We have

$$\frac{2}{\pi}\left|\int_0^\infty \cos(\tau u)\ \mathrm{Re}\left[K_0\left(4e^{\pi i/4}\sqrt{x\cosh u}\right)\right]\ du\right| \le \frac{2}{\pi}\int_0^\infty K_0\left(2\sqrt{2x\cosh u}\right)\ du$$

$$\le \frac{4}{\pi}\int_0^\infty K_0\left(2\sqrt{2x}\ \cosh u\right)\ du = \frac{4}{\pi}\int_0^\infty\int_0^\infty \exp\left(-\sqrt{2x}\ \cosh u\ \cosh v\right)$$

$$\times\ \exp\left(-\sqrt{2x}\ \cosh u\ \cosh v\right)\ dudv \le \frac{4}{\pi}\,K_0^2\left(\sqrt{2x}\right)$$

and the result follows.

□

Remark 1. Inequality (1.25) is a particular case of the inequality (1.23) with $\delta = 0$.

Employing the Mellin-Barnes representation (1.9) of the kernel in (1.1), which we denote by

$$\Psi_\tau(x) = \frac{K_{i\tau}(2\sqrt{2x})}{\sinh(\pi\tau/2)}\ \mathrm{Im}\left[J_{i\tau}(2\sqrt{2x})\right],$$

we will derive an ordinary differential equation, whose solution is $\Psi_\tau(x)$. Precisely, it is given by

Lemma 2. *The kernel $\Psi_\tau(x)$ is a fundamental solution of the following fourth order differential equation with variable coefficients*

$$x^2\frac{d^4\Psi_\tau}{dx^4} + 5x\frac{d^3\Psi_\tau}{dx^3} + (4+\tau^2)\frac{d^2\Psi_\tau}{dx^2} + 16\Psi_\tau = 0. \tag{1.26}$$

Proof. Recalling the Stirling asymptotic formula for the gamma-function [2], Vol. I, we see that for each $\tau \in \mathbb{R}$ the integrand in (1.9) behaves as ($s = \gamma + it$)

$$\frac{\Gamma(s/2)\Gamma((s+i\tau)/2)\Gamma((s-i\tau)/2)}{\Gamma((1-s)/2)} = e^{-\pi|t|/2}|t|^{2\gamma-3/2}, \quad |t| \to \infty.$$

This argument allows to differentiate repeatedly with respect to x under the integral sign in (1.9). Hence with the reduction formula for the gamma-function we obtain

$$x\frac{d}{dx}x\frac{d}{dx}\,\Psi_\tau = -\frac{1}{16\pi i\sqrt{\pi}}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{s^2\Gamma(s/2)\Gamma((s+i\tau)/2)\Gamma((s-i\tau)/2)}{\Gamma((1-s)/2)}x^{-s}ds$$

$$= -\tau^2\Psi_\tau(x) - \frac{1}{4\pi i\sqrt{\pi}}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(s/2)\Gamma(1+(s+i\tau)/2)\Gamma(1+(s-i\tau)/2)}{\Gamma((1-s)/2)}x^{-s}ds$$

$$= -\tau^2\Psi_\tau(x) - \frac{1}{4\pi i\sqrt{\pi}}\int_{2+\gamma-i\infty}^{2+\gamma+i\infty}\frac{\Gamma(s/2-1)\Gamma((s+i\tau)/2)\Gamma((s-i\tau)/2)}{\Gamma((3-s)/2)}x^{2-s}ds$$

$$= -\tau^2\Psi_\tau(x) - \frac{1}{\pi i\sqrt{\pi}}\int_{2+\gamma-i\infty}^{2+\gamma+i\infty}\frac{\Gamma(s/2)\Gamma((s+i\tau)/2)\Gamma((s-i\tau)/2)}{(s-2)(1-s)\Gamma((1-s)/2)}x^{2-s}ds.$$

Differentiating again twice both sides of the latter equality and using (1.9), we find

$$\frac{d^2}{dx^2}\left(x\frac{d}{dx}\right)^2\Psi_\tau = -\tau^2\frac{d^2\Psi_\tau}{dx^2} - 16\Psi_\tau(x).$$

Thus after fulfilling the differentiation in the left-hand side of the latter equality, we arrive at (1.26). Lemma 2 is proved.

□

Finally in this section we note that the obtained inequalities and integral representations of the kernel in (1.1) will be used in the sequel to study the boundedness, compactness and invertibility of the index transforms (1.1), (1.2).

2. Boundedness and compactness in Lebesgue's spaces

We begin, introducing the following weighted L_1- space

$$L^0 \equiv L_1\left(\mathbb{R}_+;\; K_0^2(\sqrt{2x})dx\right) := \left\{f : \int_0^\infty K_0^2(\sqrt{2x})|f(x)|dx < \infty\right\}. \tag{2.1}$$

In particular, as we will show below, it contains spaces $L_{\nu,p}(\mathbb{R}_+)$ for some $\nu \in \mathbb{R},\ 1 \le p \le \infty$ with the norms

$$||f||_{\nu,p} = \left(\int_0^\infty x^{\nu p-1}|f(x)|^p dx\right)^{1/p} < \infty, \tag{2.2}$$

$$||f||_{\nu,\infty} = \text{ess sup}_{x\ge 0}|x^\nu f(x)| < \infty.$$

When $\nu = \frac{1}{p}$ we obtain the usual norm in L_p denoted by $||\ ||_p$.

Lemma 3. *Let* $\nu < 1, 1 \le p \le \infty,\ q = \frac{p}{p-1}$. *Then the embedding holds*

$$L_{\nu,p}(\mathbb{R}_+) \subseteq L^0 \tag{2.3}$$

and

$$||f||_{L^0} \le 2^{-2(1/p+\nu)}\left[\frac{\Gamma^{1/q}\left(2q(1-\nu)\right)}{(2q)^{2(1-\nu)}}\,B(1-\nu,\,1-\nu)\right]^2 ||f||_{\nu,p},\ 1 < p \le \infty, \tag{2.4}$$

$$||f||_{L^0} \le \sup_{x\ge 0}\left[K_0^2(\sqrt{2x})x^{1-\nu}\right] \ ||f||_{\nu,1}. \tag{2.5}$$

Proof. In fact, with the definition of the norm (2.1) and the Hölder inequality we obtain

$$||f||_{L^0} = \int_0^\infty K_0^2(\sqrt{2x})|f(x)|dx \le \left(\int_0^\infty K_0^{2q}(\sqrt{2x})x^{(1-\nu)q-1}dx\right)^{1/q} ||f||_{\nu,p},\ q=\frac{p}{p-1} \tag{2.6}$$

and the latter integral via asymptotic behavior of the Macdonald function (1.5), (1.6), (1.7) converges for $\nu<1$. Hence integral (1.8) and the generalized Minkowski inequality yield

$$\left(\int_0^\infty K_0^{2q}(\sqrt{2x})x^{(1-\nu)q-1}dx\right)^{1/q} = \left(\int_0^\infty x^{(1-\nu)q-1}\left(\int_0^\infty e^{-\sqrt{2x}\ \cosh u}du\right)^{2q}dx\right)^{1/q}$$

$$\le \left(\int_0^\infty \left(\int_0^\infty x^{(1-\nu)q-1}e^{-2q\sqrt{2x}\ \cosh u}dx\right)^{1/q}du\right)^2$$

$$= 2^{2(\nu-1/p)}(2q)^{4(\nu-1)}\Gamma^{2/q}\left(2q(1-\nu)\right)\left(\int_0^\infty \frac{du}{\cosh^{2(1-\nu)}u}\right)^2.$$

Calculating the integral with the hyperbolic function via relation (2.4.4.4) in [3], Vol. I, we come up with the estimate (2.4). For the case $p=1$ we end up immediately with (2.5), using (2.6), where the supremum is finite via the condition $\nu<1$. Thus the embedding (2.3) is established and Lemma 3 is proved. □

Theorem 1. *The index transform* (1.1) *is well-defined as a bounded operator from* L^0 *into the space* $C_0(\mathbb{R})$ *of bounded continuous functions vanishing at infinity. Besides, the following composition representation holds*

$$(Ff)(\tau) = \left(F_c(\mathscr{K}_0 f)(\cosh t)\right)(\tau), \tag{2.7}$$

where the Fourier cosine transform F_c *is defined by* (1.14) *and*

$$(\mathscr{K}_0 f)(x) = -\sqrt{\frac{2}{\pi}}\int_0^\infty \mathrm{Re}\left[K_0\left(4e^{\pi i/4}\sqrt{xt}\right)\right]\ f(t)dt \tag{2.8}$$

is the operator of the Meijer type K- transform (cf. [4]*).*

Proof. In fact, the inequality (1.25) implies

$$||Ff||_{C_0(\mathbb{R})} = \sup_{\tau\in\mathbb{R}}|(Ff)(\tau)| \le \frac{4}{\pi}\int_0^\infty K_0^2(\sqrt{2x})|f(x)|dx = \frac{4}{\pi}||f||_{L^0} < \infty,$$

which means that the operator (1.1) is well-defined and the integral converges absolutely and uniformly with respect to $\tau\in\mathbb{R}$. Thus $(Ff)(\tau)$ is continuous. On the other hand, recalling (1.15) and Corollary 3, we derive

$$|(Ff)(\tau)| \le \frac{2}{\pi}\int_0^\infty\int_0^\infty K_0\left(2\sqrt{2x\cosh t}\right)|f(x)|dxdt \le \frac{4}{\pi}||f||_{L^0} < \infty.$$

Hence in view of Fubini's theorem one can invert the order of integration in the corresponding iterated integral and arrive at the composition (2.7). Moreover, the previous estimate says that $(\mathscr{K}_0 f)(\cosh t) \in L_1(\mathbb{R})$. Consequently, $(Ff)(\tau)$ vanishes at infinity owing to the Riemann-Lebesgue lemma. □

Corollary 4. *The operator* (1.1) $F : L_{\nu,p}(\mathbb{R}_+) \to L_p(\mathbb{R}),\ p \geq 2,\ \nu < 1$ *is bounded and*

$$||Ff||_{L_p(\mathbb{R})} \leq \frac{\pi^{1/p-1}}{2}\, q^{2(\nu-1)} \left[\Gamma(q(1-\nu))\right]^{2/q} B(1-\nu,\, 1-\nu)\, ||f||_{\nu,p}, \quad q = \frac{p}{p-1}. \tag{2.8}$$

Proof. Indeed, taking the composition (2.7), we appeal to the Hausdorff-Young inequality for the Fourier cosine transform (1.14) (cf. [5], Theorem 74)

$$||F_c f||_{L_q(\mathbb{R})} \leq (2\pi)^{1/q-1/2} ||f||_{L_p(\mathbb{R})},\ 1 < p \leq 2,\ q = \frac{p}{p-1},$$

we find

$$||Ff||_{L_p(\mathbb{R})} \leq \sqrt{2}\, \pi^{1/p-1/2} \left(\int_0^\infty |(\mathscr{K}_0 f)(\cosh t)|^q dt \right)^{1/q}. \tag{2.9}$$

Hence by the generalized Minkowski and Hölder inequalities with relation (2.16.2.2) from [3], Vol. II we obtain

$$\sqrt{2}\, \pi^{1/p-1/2} \left(\int_0^\infty |(\mathscr{K}_0 f)(\cosh t)|^q dt \right)^{1/q} \leq 2\pi^{1/p-1} \int_0^\infty |f(x)| \left(\int_0^\infty K_0^q \left(2\sqrt{2x}\, \cosh(t/2) \right) dt \right)^{1/q} dx$$

$$\leq 2^{1/q+1} \pi^{1/p-1} \int_0^\infty \int_0^\infty |f(x)| \left(\int_0^\infty e^{-2q\sqrt{2x}\cosh t \cosh u} dt \right)^{1/q} du dx$$

$$= 2^{1/q+1} \pi^{1/p-1} \int_0^\infty \int_0^\infty |f(x)|\, K_0^{1/q} \left(2q\sqrt{2x} \cosh u \right) du dx$$

$$\leq 2^{1/q+1} \pi^{1/p-1} ||f||_{\nu,p} \int_0^\infty \left(\int_0^\infty x^{(1-\nu)q-1} K_0 \left(2q\sqrt{2x}\, \cosh u \right) dx \right)^{1/q} du$$

$$= 2^{\nu} \pi^{1/p-1} q^{2(\nu-1)} \left[\Gamma(q(1-\nu))\right]^{2/q} ||f||_{\nu,p} \int_0^\infty \frac{du}{\cosh^{2(1-\nu)} u}$$

$$= 2^{-\nu} \pi^{1/p-1} q^{2(\nu-1)} \left[\Gamma(q(1-\nu))\right]^{2/q} B(1-\nu,\, 1-\nu)\, ||f||_{\nu,p}.$$

Consequently, combining with (2.9), we get (2.8). □

Now we investigate the compactness of the operator (1.1).

Theorem 2. *The operator* (1.1) $F : L_{\nu,p}(\mathbb{R}_+) \to L_q(\mathbb{R}),\ 1 < p \leq 2, \nu < 1,\ q = p/(p-1)$ *is compact*.

Proof. The proof is based on approximation of the operator (1.1) by a sequence of compact operators of a finite rank with continuous kernels of compact support. But to achieve this goal, it is sufficient to verify the following Hilbert-Schmidt-type condition

$$\int_0^\infty \int_{-\infty}^\infty |\Psi_\tau(x)|^q x^{(1-\nu)q-1} d\tau dx < \infty. \tag{2.10}$$

In fact, similarly as above we recall (1.15) and the generalized Minkowski inequality to deduce

$$\left(\int_0^\infty \int_{-\infty}^\infty |\Psi_\tau(x)|^q x^{(1-\nu)q-1} d\tau dx \right)^{1/q}$$

$$\leq 2^{1/2+1/p}\, \pi^{1/q-1/2} \left(\int_0^\infty x^{(1-\nu)q-1} \left(\int_0^\infty K_0^p (2\sqrt{2x} \cosh t) dt \right)^{1/(p-1)} dx \right)^{1/q}$$

$$\leq 2^{1/2+1/p}\, \pi^{1/q-1/2} \left(\int_0^\infty x^{(1-\nu)q-1} \left(\int_0^\infty K_0^{1/p} (2p\sqrt{2x} \cosh t)\, dt \right)^q dx \right)^{1/q}$$

$$\leq 2^{1/2+1/p}\,\pi^{1/q-1/2}\int_0^\infty\left(\int_0^\infty x^{(1-\nu)q-1}\,K_0^{1/(p-1)}(2p\sqrt{2x}\cosh t)\,dx\right)^{1/q}dt$$

$$= 2^{\nu-3/2}\,\pi^{1/q-1/2}p^{2(\nu-1)}B(1-\nu,\,1-\nu)$$

$$\times\left(\int_0^\infty x^{2(1-\nu)q-1}\,K_0^{1/(p-1)}(x)\,dx\right)^{1/q}\leq 2^{\nu-3/2}\,\pi^{1/q-1/2}p^{2(\nu-1)}B(1-\nu,\,1-\nu)$$

$$\times\left(\int_0^\infty du\left(\int_0^\infty x^{2(1-\nu)q-1}e^{-x\cosh u/(p-1)}dx\right)^{p-1}\right)^{1/p}=2^{-\nu+1/2-2/p}\,\pi^{1/q-1/2}q^{2(\nu-1)}\Gamma^{1/q}(2q(1-\nu))$$

$$\times B(1-\nu,\,1-\nu)B^{1/p}(p(1-\nu),\,p(1-\nu))<\infty.$$

$\square$

Another representation of the transform (1.1) can be given via the Parseval equality for the Mellin transform (1.13) and the Mellin-Barnes integral representation (1.9). In fact, an immediate consequence of Theorem 87 in [5] and Stirling's asymptotic formula for the gamma-function is

Theorem 3. *Let* $f\in L_{1-\nu,p}(\mathbb{R}_+),\ 1<p\leq 2$. *Then for all* $\tau\in\mathbb{R}$

$$(Ff)(\tau)=-\frac{1}{16\pi i\sqrt{\pi}}\int_{\nu-i\infty}^{\nu+i\infty}\frac{\Gamma(s/2)\Gamma((s+i\tau)/2)\Gamma((s-i\tau)/2)}{\Gamma((1-s)/2)}f^*(1-s)ds. \tag{2.11}$$

Finally in this section we investigate the existence and boundedness of the operator (1.2), which is the adjoint of (1.1). In fact, it follows from the general operator theory. However, we will prove it directly, getting an explicit estimation of its norm. Assuming $g(\tau)\in L_p(\mathbb{R}),\ 1<p\leq 2$ and recalling the asymptotic formula (1.5) for the Macdonald function, we find that for each $x>0$ the function $\mathrm{Re}\left[K_0\left(4e^{\pi i/4}\sqrt{x\cosh t}\right)\right]\in L_p(\mathbb{R}),\ 1<p\leq 2$. Hence via the Parseval theorem for the Fourier transform (cf. [5], Theorem 75) and equality (1.15), operator (1.2) can be written as

$$(Gg)(x)=-\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\mathrm{Re}\left[K_0\left(4e^{\pi i/4}\sqrt{x\cosh t}\right)\right](\mathscr{F}g)(t)dt,\ x>0, \tag{2.12}$$

where $(\mathscr{F}g)(t)\in L_q(\mathbb{R}),\ q=\frac{p}{p-1}$ is the Fourier transform of g

$$(\mathscr{F}g)(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(t)e^{ixt}dt \tag{2.13}$$

and the integral converges in the L_p-sense.

Theorem 4. *Let* $g\in L_p(\mathbb{R}),\ 1<p\leq 2$. *Then operator* (1.2) *is well-defined and for all* $x>0$

$$|(Gg)(x)|\leq 2^{1/(4p)-1}p^{-1/(2p)}x^{-1/(4p)}\,B\left(\frac{1}{4p},\,\frac{1}{4p}\right)||g||_{L_p(\mathbb{R})}. \tag{2.14}$$

Proof. Indeed, taking (2.12), we recall the Hölder inequality, the Hausdorff-Young inequality for the Fourier transform (2.13) [5] and the generalized Minkowski inequality to obtain

$$|(Gg)(x)|\leq\sqrt{\frac{2}{\pi}}\left(\int_{-\infty}^{\infty}K_0^p(2\sqrt{2x}\cosh(t/2))dt\right)^{1/p}||\mathscr{F}g||_{L_q(\mathbb{R})}\leq 2\pi^{1/q-1}||g||_{L_p(\mathbb{R})}$$

$$\times\int_0^\infty\left(\int_{-\infty}^{\infty}e^{-p\sqrt{2x}\,t^2\ \cosh u}dt\right)^{1/p}du=\ 2^{1-1/(4p)}p^{-1/(2p)}x^{-1/(4p)}\ ||g||_{L_p(\mathbb{R})}\int_0^\infty\frac{du}{\cosh^{1/(2p)}u}$$

$$=2^{1/(4p)-1}p^{-1/(2p)}x^{-1/(4p)}\,B\left(\frac{1}{4p},\,\frac{1}{4p}\right)||g||_{L_p(\mathbb{R})},$$

which proves (2.14).

$\square$

Theorem 5. *The operator* $G : L_p(\mathbb{R}) \to L_{\nu,r}(\mathbb{R}_+),\ 1 < p \le 2,\ q = p/(p-1), r \ge 1,\ \nu > 0$ *is bounded and*

$$||Gg||_{\nu,r} \le 2^{\nu-1+1/r-2/p}\pi^{1/q-1}\frac{\Gamma^{1/r}(2\nu r)\Gamma^{2/p}(\nu p)}{r^{2\nu}\Gamma^{1/p}(2\nu p)}\, B(\nu,\ \nu)\ ||g||_{L_p(\mathbb{R})},$$

where p, r have no dependence.

Proof. Again with (2.12), the generalized Minkowski, Hölder inequalities and the Hausdorff-Young inequality for the Fourier transform (2.13) we find

$$||Gg||_{\nu,r} \le \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}|(\mathscr{F}g)(t)|\left(\int_0^\infty x^{\nu r-1}K_0^r(2\sqrt{2x}\cosh(t/2))dx\right)^{1/r}dt$$

$$\le 2^{1/p}\sqrt{\frac{2}{\pi}}||\mathscr{F}g||_{L_q(\mathbb{R})}\left(\int_{-\infty}^{\infty}\left(\int_0^\infty x^{\nu r-1}K_0^r(2\sqrt{2x}\cosh t)dx\right)^{p/r}dt\right)^{1/p}$$

$$\le 2^{1-3\nu+1/r}\pi^{1/q-1}||g||_{L_p(\mathbb{R})}\left(\int_{-\infty}^{\infty}\frac{dt}{\cosh^{2\nu p}t}\right)^{1/p}\left(\int_0^\infty x^{2\nu r-1}K_0^r(x)dx\right)^{1/r}$$

$$= 2^{1-\nu+1/r-2/p}\pi^{1/q-1}\, B^{1/p}(\nu p,\ \nu p)\left(\int_0^\infty x^{2\nu r-1}K_0^r(x)dx\right)^{1/r}||g||_{L_p(\mathbb{R})}$$

$$\le 2^{1-\nu+1/r-2/p}\pi^{1/q-1}\, B^{1/p}(\nu p,\ \nu p)\int_0^\infty\left(\int_0^\infty x^{2\nu r-1}e^{-xr\cosh u}dx\right)^{1/r}du\ ||g||_{L_p(\mathbb{R})}$$

$$= 2^{1-\nu+1/r-2/p}\pi^{1/q-1}r^{-2\nu}\Gamma^{1/r}(2\nu r)B^{1/p}(\nu p,\ \nu p)\ ||g||_{L_p(\mathbb{R})}\int_0^\infty\frac{du}{\cosh^{2\nu}u}$$

$$= 2^{\nu-1+1/r-2/p}\pi^{1/q-1}r^{-2\nu}\Gamma^{1/r}(2\nu r)B^{1/p}(\nu p,\ \nu p)B(\nu,\ \nu)\ ||g||_{L_p(\mathbb{R})}.$$

$\square$

3. Inversion theorems

The composition representation (2.11) and the properties of the Fourier and Mellin transforms are key ingredients to prove the inversion theorem for the index transform (1.1) $(Ff)(\tau)$. Namely, we have

Theorem 6. *Let* $f(t) \in L_{1-\nu,p}(\mathbb{R}_+)\cap L_1((1,\infty);tdt),\ 1 < p \le 2, 0 < \nu < 1,\ q = p/(p-1)$, *and let the Mellin transform* (1.11) *satisfy the condition* $f^*(1) = 0$. *If, besides,* $\tau e^{\pi\tau}F(\tau) \in L_1(\mathbb{R}_+)$, *then for all* $x > 0$ *the following inversion formula holds*

$$f(x) = -\frac{4}{\pi}\frac{d}{dx}\int_0^\infty\frac{\tau(Ff)(\tau)}{\cosh(\pi\tau/2)}K_{i\tau}\left(2\sqrt{2x}\right)\ \mathrm{Re}\, J_{i\tau}\left(2\sqrt{2x}\right)d\tau. \tag{3.1}$$

Proof. In fact, since $f \in L_{1-\nu,p}(\mathbb{R}_+)$ then by virtue of Theorem 86 in [5] its Mellin transform $f^*(s) \in L_q(1-\nu-i\infty, 1-\nu+i\infty)$. Hence it is not difficult to verify with the use of Hölder's inequality that integral (2.11) converges absolutely. Moreover, taking the Fourier cosine transform (1.14) of both sides of this equality, we change the order of integration in its right-hand side by Fubini's theorem. Indeed, this is because the absolute convergence of the corresponding iterated integral can be immediately verified, employing inequalities (1.18), (1.19). Then, recalling (1.16), we come up with the equality

$$\int_0^\infty (Ff)(\tau)\cos(\tau u)d\tau = -\frac{1}{8i\sqrt{\pi}}\int_{\nu-i\infty}^{\nu+i\infty}\frac{\Gamma(s/2)\Gamma(s)}{\Gamma((1-s)/2)}f^*(1-s)\,(2\cosh u)^{-s}ds. \tag{3.2}$$

Further, differentiating both sides of (3.2) with respect to u, which is definitely allowed under conditions of the theorem, we obtain

$$\int_0^\infty \tau(Ff)(\tau)\sin(\tau u)d\tau = -\frac{\tanh u}{4i\sqrt{\pi}}\int_{\nu-i\infty}^{\nu+i\infty}\frac{\Gamma(1+s/2)\Gamma(s)}{\Gamma((1-s)/2)}f^*(1-s)\,(2\cosh u)^{-s}\,ds. \tag{3.3}$$

Meanwhile, after the substitution $p=\cosh u$ in (3.4), we find

$$-\frac{2}{\sqrt{\pi}}\int_0^\infty \tau(Ff)(\tau)\frac{\sin\left(\tau\log\left(p+\sqrt{p^2-1}\right)\right)}{\sqrt{p^2-1}}d\tau = \frac{1}{2p\pi i}\int_{\nu-i\infty}^{\nu+i\infty}\frac{\Gamma(1+s/2)\Gamma(s)}{\Gamma((1-s)/2)}f^*(1-s)\,(2p)^{-s}\,ds. \tag{3.4}$$

In the meantime, relations (2.16.2.2), (2.16.6.1) in [3], Vol. II give the representation

$$\frac{\sin\left(\tau\log\left(p+\sqrt{p^2-1}\right)\right)}{\sqrt{p^2-1}} = \frac{\sinh(\pi\tau)}{4\pi^2 i}$$
$$\times\int_{\nu-i\infty}^{\nu+i\infty}\Gamma(s)\,\Gamma\left(\frac{1-s+i\tau}{2}\right)\Gamma\left(\frac{1-s-i\tau}{2}\right)(2p)^{-s}ds,\ 0<\nu<1. \tag{3.5}$$

Hence, substituting the latter expression in the left-hand side of (3.4) and changing the order of integration with the use of the condition $\tau e^{\pi\tau}F(\tau)\in L_1(\mathbb{R}_+)$, we derive after simple changes of variables

$$\frac{1}{2\pi\sqrt{\pi}}\int_{1-\nu-i\infty}^{1-\nu+i\infty}\Gamma(1-w)(2p)^w\int_0^\infty \tau(Ff)(\tau)\Gamma\left(\frac{w+i\tau}{2}\right)\Gamma\left(\frac{w-i\tau}{2}\right)d\tau\,dw$$
$$=\int_{-\nu-i\infty}^{-\nu+i\infty}\frac{\Gamma(1-s/2)\Gamma(1-s)}{\Gamma((1+s)/2)}f^*(1+s)\,(2p)^s\,\frac{ds}{s}. \tag{3.6}$$

On the other hand, under conditions of the theorem $f^*(1+s)$ is analytic and bounded in the strip $-\nu<\mathrm{Re}s<1$. In fact, it follows from the estimate $(s=\mu+iy)$

$$|f^*(1+s)|\le\int_0^1|f(t)|t^\mu dt+\int_1^\infty|f(t)|t^\mu dt\le\left(\int_0^1|f(t)|^p t^{(1-\nu)p-1}dt\right)^{1/p}\left(\int_0^1 t^{(\mu+\nu)q-1}dt\right)^{1/q}$$
$$+\int_1^\infty|f(t)|t^\mu dt\le(q(\mu+\nu))^{-1/q}||f||_{1-\nu,p}+\int_1^\infty|f(t)|\,tdt<\infty.$$

Therefore, since the integrand in the right-hand side of (3.6) tends to zero when $|\mathrm{Im}s|\to\infty$ in this strip and $f^*(1)=0$, one can shift the contour to the right via the Cauchy theorem, integrating over the vertical line $(1-\nu-i\infty,1-\nu+i\infty)$. Hence,

$$\frac{1}{2\pi\sqrt{\pi}}\int_{1-\nu-i\infty}^{1-\nu+i\infty}\Gamma(1-w)(2p)^w\int_0^\infty \tau(Ff)(\tau)\Gamma\left(\frac{w+i\tau}{2}\right)\Gamma\left(\frac{w-i\tau}{2}\right)d\tau\,dw$$
$$=\int_{1-\nu-i\infty}^{1-\nu+i\infty}\frac{\Gamma(1-s/2)\Gamma(1-s)}{\Gamma((1+s)/2)}f^*(1+s)\,(2p)^s\,\frac{ds}{s}$$

and the uniqueness theorem for the inverse Mellin transform (1.12) of integrable functions [5] implies

$$\frac{1}{2\pi\sqrt{\pi}}\int_0^\infty \tau(Ff)(\tau)\Gamma\left(\frac{s+i\tau}{2}\right)\Gamma\left(\frac{s-i\tau}{2}\right)d\tau=\frac{\Gamma(1-s/2)}{\Gamma((1+s)/2)}\frac{f^*(1+s)}{s},\ s\in(1-\nu-i\infty,1-\nu+i\infty).$$

Thus,

$$\frac{f^*(1+s)}{s}=\frac{1}{2\pi\sqrt{\pi}}\int_0^\infty \tau(Ff)(\tau)\frac{\Gamma((1+s)/2)}{\Gamma(1-s/2)}\Gamma\left(\frac{s+i\tau}{2}\right)\Gamma\left(\frac{s-i\tau}{2}\right)d\tau, \tag{3.7}$$

Further, employing the relation (8.4.23.11) in [3], Vol. III, we find the value of the Mellin-Barnes integral

$$\frac{1}{2\pi i}\int_{1-\nu-i\infty}^{1-\nu+i\infty}\frac{\Gamma((1+s)/2)}{\Gamma(1-s/2)}\Gamma\left(\frac{s+i\tau}{2}\right)\Gamma\left(\frac{s-i\tau}{2}\right)sx^{-s}ds=-\frac{8\sqrt{\pi}}{\cosh(\pi\tau/2)}$$
$$\times\, x\frac{d}{dx}\left[K_{i\tau}\left(2\sqrt{2x}\right)\ \mathrm{Re}\, J_{i\tau}\left(2\sqrt{2x}\right)\right],\ x>0. \tag{3.8}$$

Hence, taking the inverse Mellin transform in (3.7), changing the order of integration by Fubini's theorem, which is permitted by the imposed conditions, we arrive at the inversion formula (3.1), completing the proof of Theorem 6.

□

Considering the index transform (1.2), it has

Theorem 7. *Let $g(z/i)$ be an even analytic function in the strip $D=\{z\in\mathbb{C}:\ |\mathrm{Re}z|<\alpha<1\},\ g(0)=g'(0)=0,\ g(z/i)=o(1),\ |\mathrm{Im}z|\to\infty$ uniformly in D and $g(x)\in L_1(\mathbb{R})$. Then, if the index transform (1.2) satisfies the condition $x^{-\gamma}d(Gg)/dx\in L_1(0,1),\ 1/2<\gamma<1$, the following inversion formula, which is written in terms of the Stieltjes integral holds valid for all $x\in\mathbb{R}$*

$$g(x)=\frac{4}{\pi}\,x\sinh\left(\frac{\pi x}{2}\right)\int_0^{\infty}K_{ix}\left(2\sqrt{2y}\right)\ \mathrm{Re}\, J_{ix}\left(2\sqrt{2y}\right)d(Gg)(y). \tag{3.9}$$

Proof. Indeed, substituting in (1.2) the expression of its kernel in terms of the Mellin - Barnes integral (1.9), we change the order of integration by Fubini's theorem via the condition $g(x)\in L_1(\mathbb{R})$ and the estimate

$$\int_{-\infty}^{\infty}|g(\tau)|\ \int_{\gamma-i\infty}^{\gamma+i\infty}\left|\frac{\Gamma(s/2)\Gamma((s+i\tau)/2)\Gamma((s-i\tau)/2)}{\Gamma((1-s)/2)}ds\right|d\tau$$
$$\le B(\gamma/2,\gamma/2)\int_{-\infty}^{\infty}|g(\tau)|d\tau\ \int_{\gamma-i\infty}^{\gamma+i\infty}\left|\frac{\Gamma(s/2)\Gamma(s)}{\Gamma((1-s)/2)}ds\right|<\infty,\ \gamma>0.$$

Hence

$$(Gf)(x)=-\frac{1}{16\pi i\sqrt{\pi}}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(s/2)}{\Gamma((1-s)/2)}x^{-s}\int_{-\infty}^{\infty}\Gamma\left(\frac{s+i\tau}{2}\right)\Gamma\left(\frac{s-i\tau}{2}\right)g(\tau)\,d\tau ds$$

and since $(Gf)(x)x^{\gamma-1}\in L_1(\mathbb{R}_+)$, which can be verified by moving the contour $(\gamma-i\infty,\ \gamma+i\infty)$ in the right-hand side of the previous equality to the right, the Mellin transform $(Gg)^*(s)$ exists (see [5]) and can be represented reciprocally in the form

$$(Gg)^*(s)=-\frac{1}{8\sqrt{\pi}}\ \frac{\Gamma(s/2)}{\Gamma((1-s)/2)}\int_{-\infty}^{\infty}\Gamma\left(\frac{s+i\tau}{2}\right)\Gamma\left(\frac{s-i\tau}{2}\right)g(\tau)\,d\tau. \tag{3.10}$$

Meanwhile, the Stirling asymptotic formula for the gamma function yields $(s=\gamma+iu)$

$$\frac{\Gamma((1-s)/2)}{\Gamma(s/2)}=O\left(|s|^{1/2-\gamma}\right),\ |s|\to\infty. \tag{3.11}$$

Moreover, from the definition of the index transform (1.2) and integral representation (1.9) of its kernel it is not difficult to verify that $(Gg)(x)$ is differentiable and $(Gg)'(x)x^{\gamma},\ \gamma>1/2$ vanishes at infinity. Hence, integrating by parts, we write the Mellin transform $(Gg)^*(s)$ as

$$(Gg)^*(s)=\int_0^{\infty}(Gg)(x)x^{s-1}dx=-\frac{1}{s}\int_0^{\infty}(Gg)'(x)x^{s}dx,\ \mathrm{Re\ s}=\gamma>0, \tag{3.12}$$

we see that under condition $(Gg)'(x)x^{-\gamma}\in L_1(0,1),\ 1/2<\gamma<1$

$$2^{s}\ \frac{\Gamma((1-s)/2)}{\Gamma(s/2)}(Gg)^*(s)=-2^{s-1}\ \frac{\Gamma((1-s)/2)}{\Gamma(1+s/2)}\int_0^{\infty}(Gg)'(x)x^{s}dx\in L_1(\gamma-i\infty,\ \gamma+i\infty)$$

is analytic in the vertical strip $-\gamma < \mathrm{Re}s < \gamma$ and via (3.11) tends to zero when $|\mathrm{Im}s| \to \infty$ uniformly in the strip. Appealing to relation (2.16.2.2) in [3], Vol. II, the inverse Mellin transform (1.12) implies from (3.10)

$$\frac{1}{i\sqrt{\pi}} \int_{\gamma-i\infty}^{\gamma+i\infty} 2^s \, \frac{\Gamma((1-s)/2)}{\Gamma(s/2)} (Gg)^*(s) y^{-s} ds = -\int_{-\infty}^{\infty} K_{i\tau}(y) g(\tau) \, d\tau, \; y > 0. \tag{3.13}$$

Further, writing $K_{i\tau}(y)$ in terms of the modified Bessel function of the third kind $I_{i\tau}(y)$ [2], Vol. II, we get

$$K_{i\tau}(y) = \frac{\pi}{2i\sinh(\pi\tau)} \left[I_{-i\tau}(y) - I_{i\tau}(y)\right].$$

Substituting it in the right-hand side of (3.13) and taking into account the evenness of $g(\tau)$, we find the equality

$$\frac{1}{i\sqrt{\pi}} \int_{\gamma-i\infty}^{\gamma+i\infty} 2^s \, \frac{\Gamma((1-s)/2)}{\Gamma(s/2)} (Gg)^*(s) y^{-s} ds = -\pi i \int_{-i\infty}^{i\infty} I_z(y) g(z/i) \, \frac{dz}{\sin(\pi z)}. \tag{3.14}$$

On the other hand, according to our assumption $g(z/i)$ is analytic in the vertical strip $0 \leq \mathrm{Re}z < \alpha$, tending to zero when $|\mathrm{Im}| \to \infty$ uniformly in the strip. Thus appealing to the inequality for the modified Bessel function of the third kind (see [4], p. 93)

$$|I_z(y)| \leq I_{\mathrm{Re}z}(y) \, e^{\pi|\mathrm{Im}z|/2}, \; 0 < \mathrm{Re}z < \alpha,$$

one can move the contour to the right in the right-hand side and, in turn, to the left in the left-hand side of the equality (3.14), taking into account the analytic properties of the corresponding integrand. Hence after a simple substitution we obtain

$$\frac{1}{i\sqrt{\pi}} \int_{\gamma-i\infty}^{\gamma+i\infty} 2^{-s} \, \frac{\Gamma((1+s)/2)}{\Gamma(-s/2)} (Gg)^*(-s) y^{s} ds = -\pi i \int_{\alpha-i\infty}^{\alpha+i\infty} I_z(y) g(z/i) \, \frac{dz}{\sin(\pi z)}. \tag{3.15}$$

Multiplying both sides of (3.15) by $K_{ix}(y)y^{-1}$ and integrating over $(0,\infty)$, we interchange the order of integration in both sides by the Fubini theorem. Then employing relation (2.16.2.2) in [3], Vol. II and the value of the integral (see relation (2.16.28.3) in [3], Vol. II

$$\int_0^{\infty} K_{ix}(y) I_z(y) \frac{dy}{y} = \frac{1}{x^2+z^2},$$

we come up with the equality

$$\frac{1}{4i\sqrt{\pi}} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma((1+s)/2)}{\Gamma(-s/2)} \Gamma\left(\frac{s+ix}{2}\right) \Gamma\left(\frac{s-ix}{2}\right) (Gg)^*(-s) ds = -\pi i \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{g(z/i) \, dz}{(x^2+z^2)\sin(\pi z)}.$$

Now, taking in mind (3.8), (3.12) and the Parseval identity (1.13), the previous equality becomes

$$\frac{1}{\cosh(\pi x/2)} \int_0^{\infty} K_{ix}\left(2\sqrt{2y}\right) \, \mathrm{Re}\, J_{ix}\left(2\sqrt{2y}\right) d(Gg)(y) = \frac{i}{2} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{g(z/i) \, dz}{(x^2+z^2)\sin(\pi z)}. \tag{3.16}$$

Meanwhile, with the evenness of g we write the right-hand side of (3.16) in the form

$$\frac{i}{2} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{g(z/i) \, dz}{(x^2+z^2)\sin(\pi z)} = \frac{i}{4} \left(\int_{-\alpha+i\infty}^{-\alpha-i\infty} + \int_{\alpha-i\infty}^{\alpha+i\infty} \right) \frac{g(z/i) \, dz}{(z-ix)\, z\sin(\pi z)} = \frac{\pi g(x)}{2x\sinh(\pi x)},$$

appealing to the Cauchy formula, because $g(z/i)/(z\sin(\pi z))$ is analytic in the strip $|\mathrm{Re}z| < \alpha < 1$ and tends to zero when $|\mathrm{Im}z| \to \infty$ uniformly in the strip. Thus, combining with (3.16), we arrived at the inversion formula (3.9) and complete to proof of the theorem.

□

4. Initial value problem

The index transform (1.2) can be successfully applied to solve an initial value problem for the following fourth order partial differential equation, involving the Laplacian

$$\left[\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2+4\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)+2\right]\Delta u+16u=0, \tag{4.1}$$

where $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ is the Laplacian in $\mathbb{R}^2$. Indeed, writing equation (4.1) in polar coordinates (r,θ), we end up with the equation

$$\frac{\partial^2}{\partial r^2}\left(\left(r\frac{\partial}{\partial r}\right)^2 u+\frac{\partial^2 u}{\partial\theta^2}\right)+16u=0. \tag{4.2}$$

Lemma 4. *Let* $g(\tau)\in L_1\left(\mathbb{R};e^{\beta|\tau|}d\tau\right),\ \beta\in(0,2\pi)$. *Then the function*

$$u(r,\theta)=\int_{-\infty}^{\infty}K_{i\tau}(2\sqrt{2r})\ \mathrm{Im}\left[J_{i\tau}(2\sqrt{2r})\right]\frac{e^{\theta\tau}g(\tau)}{\sinh(\pi\tau/2)}\,d\tau \tag{4.3}$$

satisfies the partial differential equation (4.2) *on the wedge* $(r,\theta): r>0,\ 0\le\theta<\beta$, *vanishing at infinity.*

Proof. In fact, the proof follows from the direct substitution (4.3) into (4.2) and the use of Lemma 2. The necessary differentiation with respect to r and θ under the integral sign is allowed via the absolute and uniform convergence, which can be justified with the use of (1.9) and the integrability condition $g\in L_1\left(\mathbb{R};e^{\beta|\tau|}d\tau\right),\ \beta\in(0,2\pi)$ of the lemma. □

Finally, as a direct consequence of Theorem 7, we will formulate the initial value problem for equation (4.2) and give its solution.

Theorem 8. *Let*

$$g(x)=\frac{4}{\pi}\,x\sinh\left(\frac{\pi x}{2}\right)\int_0^{\infty}K_{ix}\left(2\sqrt{2y}\right)\ \mathrm{Re}\ J_{ix}\left(2\sqrt{2y}\right)dG(y)$$

and $G(y)$ *satisfy conditions of Theorem* 7. *Then* $u(r,\theta),\ r>0,\ 0\le\theta<\beta$ *by formula* (4.3) *will be a solution of the initial value problem for the partial differential equation* (4.2) *subject to the initial condition*

$$u(r,0)=G(r).$$

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